# PHYSICAL INITIAL CONDITIONS
# AND THE MACHIAN CHARACTER OF GENERAL RELATIVITY

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ABSTRACT

The usual initial conditions considered in General Relativity are shown to suffer from two defects. On the one Hand, their physical measurement is shown to involve three instants of time instead of two and, on the other hand, these initial conditions are subjected to consistency conditions, a regrettable situation according to Wigner. It is shown that these two negative features are closely related to Mach's principle. Physical initial conditions are proposed which suffer from none of these defects.

A precise definition of "Mach's Principle" is given. A particular approach to general relativity is presented involving the consideration of physical initial conditions. General Relativity is proven to be strictly Machian, provided gravitational radiation energy is considered as being part of the distribution of matter which determines the 'privileged' system of reference.
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INTRODUCTION 1: Various approaches have been considered to demonstrate the Machian properties of Einstein's General relativity. It was thought that the integral form of Einstein'e equation was the definitive solution to the question.

The present paper takes a 'frontal' approach. It deals directly with the relation between the distribution of matter and the determination of 'the inertial frames'. It demonstrates that, in General Relativity, the distribution of matter does determine the inertial frames, provided a proper generalization of the notion of 'inertial frames' is made, and provided gravitational energy is included in the distribution of matter.

We will deal in this paper with the following aspect of Mach's principle: The inertial frames are determined by the distant stars (the distribution of matter)

At the time the principle was formulated, space was considered flat and absolute, while time was considered absolute. Though Mach's principle challenges the concept of absolute space, it accepted the notion of inertial frames.

Even within these limitations, Mach's principle, as enunciated above, is not a sharp mathematical concept. Its translation into modern gravitation, with its covariant Riemannian theories, is bound to be still more vague and general. To reduce the difficulties, we will take one step at a time and will consider:

a) Newton's laws without a Machian principle
b) Newton's laws with an ad-hoc Mach's principle
c) A generalized definition of inertial frames and a corresponding general formulation of Mach's principle
d) General Relativity and Mach's principle

## NEWTON'S LAWS WITHOUT MACH'S PRINCIPLE

Newton equations being of second order, the future of a system of particles is determined if, at an initial time, we are given the masses, positions and velocities of the particles. This statement is not quite true. In addition, we must either state that the initial positions and velocities where measured in an inertial frame, or give additional informations specifying the motion of the observing frame relative to an inertial one. In case nothing is known about the relative motion of the observer's frame, we could still determine the future of the system if we were given, in addition, the accelerations of the particles. The system would then be over-determined and would be subjected to consistency conditions. According to Wigner[1] this is not a healthy sign for any theory.

## NEWTON'S LAWS WITH MACH'S PRINCIPLE.

Mach's principle states that the inertial frames are determined by the distribution of masses (the distant stars). Therefore, in a Machian theory, the movement of the observer's frames relative to inertial ones is a redundant information. The 'whereabouts' of the inertial frames could be found from the 'whereabouts' of the distribution of masses.

We therefore adopt the following precise definition of Mach's principle in a flat space, as it could have been formulated in Mach's times:

***A theory of gravitation will be called Machian if in it the evolution of the system can be uniquely determined from a regular set of initial data that includes no information on the reference system relative to which the data are referred. In particular, it need not be known if the observer's frame of reference is inertial or not.***

By a regular set of initial data we mean the values of the functions characterising the state of the system and their first time derivatives at an arbitrary initial time.

Let us apply this definition on the following example of initial data: the universe is composed of only two equal masses separated by a distance 'd'. The observed initial velocities are -v and v and are perpendicular to the line joining the two masses.

The Newtonian theoretician will say: if the observed velocities were measured by an inertial observer, the two masses would describe conics and, for particular values of the masses velocities and initial distance, the conics could be a common circle, and the distance between the two masses would then remain constant.

It could however well be that the masses, at the initial time, are at rest relatively to an inertial frame and that the observed velocities v and -v are the result of the rotation of the observer's frame. In this case, the distance between the two masses would decrease till the two masses collide. In fact there is an infinity of possible observing frames. The possible future motions are therefore infinitely numerous. According to our definition, Newton's gravitation is not Machian.

NEWTON'S GRAVITATION FITTED WITH AN AD-HOC MACHIAN PRINCIPLE ,

1We could add to Newton's theory a Machian principle asserting that the inertial frames are the ones in which the total linear and angular momentum are both zero. In such a case, the previous initial data problem would have a unique solution. The given initial data correspond to a non-zero total angular momentum. This would indicate that the reference system is rotating relatively to the inertial frames.

It is evident in this case that the two masses are initially at rest in an inertial frame and will therefore move towards each other till collision, though, in the observer's frame, they will be seen spiralling towards each other till collision. We no longer need additional information, such as initial accelerations to determine the future of the system.

Let us underline the two important following conclusions:

1) It is the addition of an ad-hoc Mach Principle that allows the equations of motion to be of second order with respect to a set of particular observers singled out by this principle.

2) It is the same addition that 'cleans' the theory by allowing arbitrary initial conditions (not subjected to consistency equations).

GENERALIZED FORMULATION OF MACH'S PRINCIPLE

1There are no global inertial frames in General Relativity. It is however possible to generalize this concept to a Riemanian space-time. In order to avoid confusion we will call the proposed frames 'privileged frames'. In our definition, the privileged frames are those in which the speed of light is constant in space, in time and in whatever direction.

Let us consider a Riemanian space

$$. ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta ,$$

the geodesics of which are the trajectories of test particles while the null geodesics give the light trajectories. (Greek indices will take the values 0,1,2,3 while latin indices take the 3 values 1,2,3)

The speed of light, at any point and in any direction, is then given by[2]

$$v(n^j) = -cg_{oo} / [g_{jo} n^j + (-g_{oo})^{1/2}]$$

in which $n^j$ is the unit 3-vector in the direction of light propagation. The privileged frames therefore will be characterized by

$g_{io} = 0$ and $g_{oo} =$ ctte. (which we will chose to be –1)

It can be shown that a test particle acted upon by gravitational forces only, will remain at rest in such frames if it is there at rest at any initial moment. The privileged frames therefore share this important property which characterizes the Newtonian inertial frames.

It is clear that the knowledge of the $g_{\alpha o}$ and the $g_{\alpha o,o}$ at an initial time $t_o$, is equivalent to knowledge about privileged frames. If $g_{io} = 0$ and $g_{oo} =$ -1 then the observer is a privileged one. If not, the knowledge of $g_{\alpha o}$ and the $g_{ij}$, and their time derivatives, is enough to determine the relation of the observer's frames to that of a privileged observer. A transformation (equ. 20) would then allow to determine the data as seen by a privileged observer.

To specify $g_{\alpha o}$ and the $g_{\alpha o,o}$ would therefore be to specify the referring frames with respect to privileged ones. In a Machian theory, these values should not be given as part of the initial data.

Our previous definition of Mach's principle can now be generalized by replacing the term 'inertial' by 'privileged'. The definition would then have a general validity. In particular, in the important case of Riemanian space-times, we adopt the following definition:

*A theory of gravitation in Riemanian space-time will be called Machian if in it the evolution of the system can be uniquely determined from a regular set of initial values for* $g_{ij}$ and $g_{ij,o}$ *( i, j =1,2,3) with no knowledge whatsoever of* $g_{\alpha o}$ and $g_{\alpha o,o}$

This definition has at least the merit of giving a practical criterion of 'Machianity'. We call these restricted initial values, physical initial values, to be distinguished from the traditionally accepted initial values that could then be called mathematical initial values.

GENERAL RELATIVITY AND MACH'S PRINCIPLE

We have seen earlier that, in the absence of knowledge about the observer's frame of reference, and in the absence of an additional ad-hoc principle, Newton's gravitation laws could not determine the future evolution of a system unless, in addition to masses, positions, velocities, and
accelerations were also given (subject to conditions).

In general relativity however, the initial data of $g_{\alpha\beta}$ and $g_{\alpha\beta,o}$ , if it satisfies consistency equations, will determine the future evolution of the system[3]

Einstein's equations are usually divided into two sets: one set of 4 equations with no second time-derivatives, which are conditions to be satisfied by the initial data (which is therefore not arbitrary) and a set of 6 equations which determine the evolution of the system.

It seems therefore that the knowledge of the state of the system at time t and t+dt is enough to determine its future evolution. This is somewhat paradoxical since Newton's gravitation, which needs three instants in the absence of knowledge on the frame of reference, is a good approximation to general relativity for very weak fields

The paradox disappears if we observe that:

1) As in the Newtonian case in which we have no knowledge concerning the whereabouts of the inertial case, in General Relativity the initial data is subjected to consistency conditions.

2) In order to measure the $g_{\alpha o}$ and the $g_{\alpha o,o}$ , the observer needs three different times. The $g_{ij}$ can be found from distance measurements between the positions of the masses and the test particles at the initial moment $t_o$ . However, the $g_{\alpha o}$ necessitate two instants since they are obtained by the measurement of the speed of light. Their time derivatives necessitate therefore three instants of observation and are indications on the acceleration of light.

3) In the Newtonian case, the ad-hoc addition of Mach's principle allows one to determine the future evolution from the given observations at time t and t+dt. Similarly, if Einstein's theory is Machian, the privileged frames should be deductible from the initial $g_{ij}$ and $g_{ij,o}$ without knowledge of the $g_{\alpha o}$ and the $g_{\alpha o,o}$ (which would give away the frame of reference) Let us underline the following conclusions:

a) If general relativity is Machian the $g_{\alpha o}$ and their time derivatives, are not part of the initial data. In this case, the 4 equations that used to be consistency conditions imposed on the initial data become 4 equations determining the $g$♋ as function of the arbitrary initial conditions ( $g_{ij}$ and $g_{ij,o}$ }. As to the remaining six equations, their needed initial data of $g_{ij}$ and $g_{ij,o}$ can now be arbitrary.

b) If general relativity is Machian, the $g_{\alpha o}$ and the $g_{\alpha o,o}$ are not part of the initial data which now contains only the $g_{ij}$ and $g_{ij,o}$ which need only two instant t and t+dt for their measurement

c) The exclusion of the $g_{\alpha o}$ and the $g_{\alpha o,o}$ from the set of initial conditions, results from physical requirements. It is to be noted how much this fact is related to Mach's principle.

The parallel with the Newtonian case is evident. Let us examine the initial value problem in general relativity. In order for general relativity to be a Machian theory, the 4 equations for the $g$♋ should determine a solution for the observer's frame which, together with the solution to the 6 other equations, would single out a unique physical solution for the evolution of the system.

The 4 equations comprise one algebraic equation determining $g^{oo}$ and three partial differential equation on the initial hypersurface $t = t_o$. The 3 partial differential equations take the form[4]

$$KT_i^o = (1/V)[P_i^j - ♎_i^j \mathrm{P}]_{;\,j} \qquad (2)$$

while the algebraic equation is ,

$$KT_o^o = (1/2)[P^2 - H^2 - \bar{R}] \qquad (3)$$

In these equations $\bar{R}$ is the scalar 3-space curvature. All operations on spatial indices (including covariant derivatives) are on the initial 3-space. $P_{ij}$ is defined by:

$$P_{ij} = [1/2V](g_{ij,o} - g_{oj;i} - g_{oj;i}) \qquad (4)$$

and $P = P_i^i$ while $H^2 = P_i^j P_j^i$ and $V^2 = (g_{oo})^{-1}$ (5)

Indices are raised with $e^{ij}$ which is the inverse of the 3- metric $g_{ij}$. We have for instance: $P_j^i = e_a^i P_j^a$ . Similarly we also have: $P_{ij} = g_{ai} P_j^a$

From (3) and (5) $g^{oo}$ can be expressed as:

$$g^{oo} = (\Lambda^2 - \Lambda_i^j \Lambda_j^i)/(2kT_o^o + \bar{R}) \qquad (6)$$

with $\Lambda_{ij} = g_{ij,o} - g_{oj;i} - g_{oi;j} = 2VP_{ij} \qquad \Lambda^2 = \Lambda_u^u$ (7)

$g_{oo}$ is obtained from $g^{oo}$ through

$$g_{oo} = e^{kl} g_{ok} g_{ol} + 1/(g^{oo}) \qquad (8)$$

while $g_{oj;i}$ is the 3-covariant derivative of the 3-vector $g_{oj}$ .

The problem of finding solutions for the $g_{oi}$ , can be divided into two steps

First step: we solve equations (2) and (6) as determining the $P_{ij}$ and $g_{oo}$ . In this step $P_{ij}$ is considered as an unknown 3-tensor which is to be expressed in function of $g^{oo}, g_{ij}, T_o^o$ and $T_i^o$ only

Second step: For each set of $P_{ij}$ and $g^{oo}$ so determined, we solve (4) for the $g_{oi}$
We will now prove that, for a given set of the $P_{ij}$ and $g^{oo}$ , all different solutions of (4) for the $g_{oi}$ , orrespond to one and the same physical solution expressed in different coordinate systems.

Let us for instance start with a particular solution of (4) for the $g_{io}$ and let us perform the following transformations: .

$\bar{x}^i = x^i - te^{iu} g_{uo}$ and $\bar{t} = t$ (9) ,
At the initial time $\bar{t} = t = 0$ , we shall have the following (see appendix):

$$\begin{aligned}
&\bar{g}^{oo} = g^{oo} \\
&\bar{P}_{ij} = P_{ij} \\
&\bar{g}_{ij} = g_{ij} \\
&\bar{g}_{io} = g_{io} \\
&\bar{T}_o^i = T_o^i \\
&\bar{g}_{ij,o} = g_{ij,o} - g_{oi;j} - g_{oj;i} = 2P_{ij}\sqrt{-g^{oo}}
\end{aligned} \qquad (10)$$

Therefore, given the initial data $g_{ij}$ and $g_{ij,o}$ , all different solutions of (2) and (3) corresponding to the same solution for $P_{ij}$ and $g_{oo}$ , represent one and the same physical solution expressed in different coordinate systems. Therefore, the multiplicity of different physical solutions can only occur from the first step that determines $P_{ij}$ , The uniqueness of the physical solution corresponding to a given $P_{ij}$ , has a definite Machian meaning.

Let us consider, for instance, an initial distribution of matter with cylindrical symmetry in an initial 3-space cylindrically symmetric. Let $Q^i$ be the Killing vector characterizing this symmetry and let us consider a particular solution for $P_{ij}$ and a particular set of $g_{io}$ corresponding to it. Let us now perform the following transformation of coordinates:

$$. x^i = \overline{x}^i + \overline{t}Q^i \qquad t = \overline{t} \qquad (11)$$

At the time t =t =0 we will have:

$$\overline{g}_{ij} = g_{ij} \quad ; \quad \overline{g}_{io} = g_{ia}Q^a \quad ; \quad \overline{g}^{oo} = g^{oo} \qquad (12)$$

$$\overline{g}_{ij,o} = g_{ij,o} + g_{ij,a}Q^a + g_{aj}Q^a_{,i} + g_{ai}Q^a_{,i}$$

which can readily be rewritten: (see appendix)

$$\overline{g}_{ij,o} = g_{ij,o} + Q_{i;j} + Q_{j,i}$$

and since $Q^i$ is a killing vector, we will have:

$$\overline{g}_{ij,o} = g_{ij,o} \qquad \overline{P}_{ij} = P_{ij}$$

This is indeed a purely Machian effect. The two observers related by the coordinate ransformation are turning one relatively to the other, though, at time t=0 their frames coincide. They observe the
same initial data corresponding to two different kinematical situations, and come up with the same physical solution.

It is clear that in the Newtonian case we would have had two different physical solutions because the same numerical data referred to inertial frames and to rotational frames give rise to two different physical situations.

The fact that this Machian effect is exhibited in the case of the existence of a Killing vector in the 3-space does not mean that the effect does not exist otherwise. However, only in that case is it possible to have the same value of $g_{ij,o}$ for the two observers.

There remains an essential difficulty which is encountered in step one: equations (2) and (6) have more than one solution for the $P_{ij}$ corresponding to the initial data of the $g_{ij}$ , and it can be shown that two different solutions of (2) and (6) determine two different physical solutions.

Strictly speaking, and though every physical solution corresponding to a given $P_{ij}$ exhibits the Machian features inherent in the multiplicity of the $g_{o\alpha}$ solutions corresponding to one same physical solution, the fact that the $g_{ij}$ and $g_{ij,o}$ determine an unique solution for the $P_{ij}$ would seem to be a non-Machian feature (according to our definition) : the regular initial data do not determine uniquely the future evolution of the system.

Let us see if it is possible and desirable to give a physical criterion that would allow to choose one, and only one, out of all possible solutions. Such a subset of solutions of Einstein equations would then become Machian. Such a procedure is always possible with any theory. However, if the Machian character is added ad-hoc and does not derive from the spirit of the theory, we could hardly say that the theory is Machian. We could, at most, say that a Machian principle has been superimposed on it.

Let us have a closer look at equation (2). We can rewrite it:

$$M^i_{j;i} = Kg^{oo(-1/2)}T^o_j$$

It expresses the divergence of the tensor field $M^i_j$ as being equal to the vector source $T^o_j$ (within a factor) with

$$M_{ij} = P_{ij} - g_{ij}(P - a)$$

in which 'a' is a constant on the hypersurface t=0 and may therefore be a function of time t.

In Electromagnetism, the divergence of field vectors is equal to their sources. The solution of such equation is given by adding to a particular solution, the general solution of the homogeneous equation obtained by equating the source to zero. In flat space, of all the solutions to the divergence equation, only one is zero at infinity and tends to zero when the source tends to zero. All other solution are obtained by adding to this one the curl of any field vector. The uniqueness of the solution can be obtained either if we know that there are no wave components in it or if we are given additional information as to the initial wave distribution.

In general relativity, the situation is similar. Either we decide to single out that solution for the $M_{ij}$ which depends solely on the matter tensor (no radiation, and that means to choose $M_{ij} = 0$ in vacuum) or to be aware that the initial data for the $g_{ij}$ and the $g_{ij,o}$ are compatible with different initial distributions of gravitational radiation. ,

We are led therefore to examine the two possible alternatives:

1) We, arbitrarily, single out the $M_{ij}$ which tends to zero with the sources and which is zero itself if the source is zero. This leads to a unique physical solution when the $g_{ij}$ and the $g_{ij,o}$ are given while the $g_{io}$ and the $g_{io.o}$ are unknown. This unique solution will be considered later. The fact that the waveless solution is unique, points to the source of multiplicity of solutions. It is due to the multiplicity of the possible initial wave configurations.

2) In addition to the regular initial data ($T_{\alpha\beta}, g_{ij}, g_{ij,o}$), we also specify just enough information on the radiation distribution, allowing us to single out a corresponding unique solution for the $M_{ij}$.In this case, general relativity is still Machian according to our definition (no $g_{io}$ and $g_{io,o}$ in the initial data)

Let us consider the case with $T_{io} = 0$. The radiation-less case $M_{ij} = 0$ can according to (15) and (16) be written:

$$P_{ij} - g_{ij}(P - a) = 0 \qquad (17) ,$$

The solution of (17) is unique and given by:
.

$$P_{ij} = (a/2)g_{ij}. \qquad (18)$$

or $$g_{ij,o} - g_{io,j} - g_{jo,i} = a(-g^{oo})^{-1/2} g_{ij} \qquad (19) ,$$

Eqs. (19) represent a set of six partial differential equations determining the three unknown $g_{io}$ (together with a seventh algebraic equation for the $g_{oo}$).

It can be shown (see appendix) that the coordinate transformation:

$$t = (-g^{oo})^{-1/2}\bar{t} \ ; \ \mathrm{x}^{\mathrm{i}} = \bar{x}^{j} - \bar{t}g_{jo}(-g^{oo})^{-1/2)} e^{ij} \qquad (20)$$

does not affect the values of the 3-metric $g_{ij}$ while it makes all the $g_{io}$ equal to zero and $g_{oo}$ equal to -1. In the new coordinates, $T_i^o = 0$ implies $T_i^o = 0$ .What follows is therefore valid for a
radiationless universe in comoving coordinates.

Equations (19) becomes:

$$g_{ij,o} = ag_{ij} \text{ or } (g_{ij,o})/(g_{ij}) = a(t)$$

the solution of which is:

$$g_{ij} = f_{ij}(x1, x2, x3)e^{\int (1/2)a(t)dt}$$

The Robertson-Walker metric is a very particular case of (21) while Godel's metric cannot be put into this form.

From the point of view of the initial data problem, the function of time in (21) is a constant at the initial time t=0. This constant can be absorbed in the function f which becomes determined by the initial data. The radiation-less condition is then expressed in the fact that the $g_{ij,o}$ must be such that $g_{ij,o} = ag_{ij}$ . This condition, if satisfied at time t=0 will remain satisfied at time t with a function of a(t)determined by the 6 evolution equations. ,

The condition $M_{ij} = o$ has a clear physical meaning. The quantity

$$8\pi J = \int \xi_i (P^{ij} - g^{ij} P) d\sigma_j \qquad (22)$$

is a conserved one if $\zeta_i$ is a Killing vector of the 3-space[5] or if $\xi_i$ is an assymptotic Killing vector in an assymptotically flat 3-space. The assymptotic Killing vectors for rotational and translational symmetry will give rise to a conserved linear and angular momentum.

The condition $M_{ij} = 0$ makes the expression in (22) equal to zero and states therefore that, in the radiationless case, the total angular momentum and the total linear momentum are zero (in a comoving privileged system). ,

We have seen that if the initial data are compatible with the radiationless conditions, then the solution given by general relativity is unique. It is clear that in the case of gravitational radiation, the knowledge of its initial distribution would restrict the multiplicity of possible solutions of equations (2)

We know[6] that gravitational waves can give rise in the metric to terms of a Schwatschild character (1-2m/r) at great distances from the waves. Bodies could therefore have a planetary motion round temporarily localized radiation. It is therefore reasonable to expect that the gravitational radiation would partake in the determination of the whereabouts of the privileged frames.

Let us consider the case of an observer who, at the initial time, in empty space, observes a 3-flat space with a 3-metric having null time derivatives. In the absence of the knowledge of the $g_{oi}$ 's and $g_{oo}$ the future of the system is indeterminate, though we know that one of the possible solutions is the Minkowski space. Let us now impose the condition that there is no gravitational energy, i.e. let us specify the simplest gravitational energy distribution: Zero energy.

Witten[7] has proven that, in the case in which there exist a space-like hypersurface that is assymptotically euclidian, the space has to be Minkowskian if the total energy is zero. This means that the physical solution is unique in that case.

Specifying the wave distribution (null) is therefore enough to determine the physical solution.

The case of spherical symmetry is a good illustration. In this case, Birkhoff's theorem ensures that, in free space, the solution is unique. This could be considered as resulting from the absence of gravitational radiation in this case.

And finally, equations (15) being three partial differential equations of first order, have a multiplicity of solution determined by three arbitrary functions. That is precisely the amount of degrees of freedom' involved in the initial distribution of gravitational radiation. It can be shown[8] that, in the immediate neighbourhood of a point, a locally Lorentz metric can be introduced with $x^1$ pointing in the direction of wave propagation, in which case all the local non-vanishing components of the Rieman tensor can be expressed in function of the arbitrary values of the 1st and 2nd time derivatives of the three functions $g_{22,}\, g_{33}$ and $g_{23}$. At the initial time these three functions and their first time-derivatives are part of the initial data. To determine the distribution of gravitational radiation the three second derivatives at initial time must also be given.

Referring to Weber's equations (7.27) to (7.30) it is easy to find out that only three components of the Rieman tensor are not known at the initial time. The others are either null or dependent on the initial conditions for the $g_{ab}$ and the $g_{ab,o}$ or, still can be expressed in function of the contracted Rieman tensor whose initial values are determined by the initial values of the matter tensor $T_{ab}$. From the same equations we have:

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$$g_{22,00} + g_{33,00} = (R_{00} + R_{22} + R_{33} - R_{11}) / 2$$

The initial value of the right hand side is determined by the initial value of the matter tensor $T_{ab}$ Therefore, the initial values of $g_{22,0}$ and $g_{33,0}$ are not independent. As a result, given the initial values of $g_{ab}$ , $g_{ab,o}$ and $T_{\mathrm{ab}}$ all the initial values of the components of the Rieman tensor depends on the initial values of $g_{22,00}$ , $g_{33,00}$ and $g_{23,00}$ subject to the fact that the sum $g_{22,00} + g_{33,00}$ is also known.

There seems to be two degrees of freedom to determine the Rieman tensor; however, we can choose at each point the direction of wave propagation which gives us, apparently, 2 more degrees of freedom or a total of 4 degrees of freedom. However, one of the degres is a coordinate effect since we can still turn the coordinate system round the $x_i$ axis without affecting the physical distribution of the wave pattern. There therefore remains three degrees of freedom that allows, once the initial conditions are determined, to specify a wave distribution.

This tends to prove that, in addition to the Machian requested data, three more arbitrary functions are involved corresponding to the amount of arbitrariness existing in the distribution of the initial gravitational radiation.

We therefore conclude that:

1) In the case of universes specified as not having gravitational waves, General Relativity is strictly Machian according to our definition of Machianity.

2) In the more general case, when gravitational waves are considered, General Relativity is still a strictly Machian theory, provided that gravitational radiation is accepted as a factor in determining the whereabouts of the privileged frames, i.e. provided we agree that initial conditions must contain the proper information allowing to choose a particular initial wave configuration which determine the $P_{ij}$ uniquely.

In cases where the mass distribution do not seem to determine the privileged frame (Goedel's solution for instance) one should look at the determination of the privileged frame by the combined effect of masses and gravitational radiation.

The radiationless vacuum case represented by flat space-time seems to contradict Mach's principle. There are in this case inertial effects that derive neither from a mass distribution nor from gravitational waves distribution. This case will be dealt with in a separate paper.

REMARK When speaking of the initial data as viewed by an observer who does not know how related he is to a privileged system of reference, the relevant data available to the observer are the
contravariant component of the metric and their time derivatives i.e. $g^{ij}$ and $g^{ij}_{,o}$ since, as is well known the 3-metric $\gamma_{ij}$ observed at an initial time is related to $g_{ij}$ by $\gamma_{ij} g^{jk} = \delta_i^k$. A more cnsistent approach would have considered the $g^{ij}$ and $g^{ij}_{,o}$ as the appropriate initial data. ,

## A P P E N D I X
## .Derivation of some formula

Let us consider the transformation

$$x^i = \overline{x}^i + \overline{t}P^i \qquad\qquad t = Q\overline{t} \qquad\qquad (1)$$

in which P and Q are independent of t.

We are interested in calculating the quantities $\overline{g}_{ij}$ , $\overline{g}_{ij,o}$ , and $\overline{g}_{oo}$ at the initial time $t_o = \overline{t}_o = 0.$
The highest time derivative we will be using being one, we need not consider terms with powers of t higher than the first (at t=0) ,

We have $\dfrac{\partial x^i}{\partial \overline{x}^j} = \delta^i_j + \overline{t}P^i_{,\overline{j}} = \delta^i_j + \overline{t}P^i_{,a}\dfrac{\partial x^a}{\partial \overline{x}^j}$

If we replace $\dfrac{\partial x^a}{\partial x^j}$ by $(\delta^a_j + \overline{t}P^a_{,\overline{j}})$ and neglecting terms in $t^2$ we will have:

$$\frac{\partial x^i}{\partial \overline{x}^j} = \delta^i_j + \overline{t}P^i_{,j}$$

Similarly $\dfrac{\partial x^i}{\partial \overline{t}} = P^i + \overline{t}\dfrac{\partial P^i}{\partial \overline{t}} = P^i + \overline{t}\dfrac{\partial P^i}{\partial x^\alpha}\dfrac{\partial x^\alpha}{\partial \overline{t}}$

Neglecting terms in $t^2$ we will have ,

$$\frac{\partial x^i}{\partial \overline{t}} = P^i + \overline{t}P^i_{,j}P^j \ ,$$

Likewise, we will have $\dfrac{\partial t}{\partial \overline{x}^i} = Q_{,i}\overline{t}$ and $\dfrac{\partial t}{\partial \overline{t}} = Q + \overline{t}Q_{,i}P^i$ .

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Grouping together the results for the transformation, we write

$$\frac{\partial x^i}{\partial \overline{x}^j} = \delta^i_j + \overline{t}P^i_{,j} \qquad\qquad \frac{\partial t}{\partial \overline{x}^j} = Q_{,j}\overline{t}$$

$$\frac{\partial x^i}{\partial \overline{t}} = P^i + tP^i_{,j}P^j \qquad\qquad \frac{\partial t}{\partial \overline{t}} = Q + \overline{t}Q_{,i}P^i \qquad\qquad (23)$$

We can write

$$\bar{g}_{ij} = g_{uv}\frac{\partial x^u}{\partial \bar{x}^i}\frac{\partial x^v}{\partial \bar{x}^j} + g_{ou}\left(\frac{\partial t}{\partial \bar{x}^i}\frac{\partial x^u}{\partial \bar{x}^j} + \frac{\partial t}{\partial \bar{x}^j}\frac{\partial x^u}{\partial \bar{x}^i}\right) + g_{oo}\frac{\partial t}{\partial \bar{x}^i}\frac{\partial t}{\partial \bar{x}^j}$$

Using equations (23) we have (neglecting $t^2$)

$$\bar{g}_{ij} = g_{ij} + \bar{t}\,(g_{uj}P^u_{,i} + g_{iv}P^v_{,j} + g_{oj}Q_{,i} + g_{oi}Q_{,j}) \qquad (24)$$

For $g_{ij,o}$ we may neglect all powers of t. Taking into account that $\bar{g}_{ij,i\bar{o}} = \bar{g}_{ij,\alpha}\dfrac{\partial x^\alpha}{\partial \bar{t}}$ we obtain

$$\bar{g}_{ij,\bar{o}} = g_{ij.o}Q + g_{ij,u}P^u + g_{uj}P^u_{,i} + g_{ui}P^u_{,j} + g_{oj}Q_{,i} + g_{oi}Q_{,j} \qquad (25)$$

Let us now show that,

$$g_{ij,u}P^u + g_{uj}P^u_{,i} + g_{ui}P^u_{,j} = P_{i;j} + P_{j;i} \qquad (26)$$

Replacing in (26) $p^U$ by $e^{ua}P_a$ , and $P^u_{,i}$ by $e^{ua}_{,i}P_a + e^{ua}P_{a.i}$ and taking into consideration that

$e^{ua}_{,i} = -g_{kl,i}e^{uk}e^{ul}$ we have therefore $P^u_{,i} = e^{ua}P_{a,i} - P_a(g_{kl,i}e^{uk}e^{al})$

Therefore $g_{uj}P^u_{,i} = P_{j,i} - P_a(g_{jl,i}e^{al})$

and similarly $g_{ui}P^u_{,j} = P_{i,j} - P_a(g_{il,j}e^{al})$

The expression $g_{ij,u}P^u + g_{uj}P^u_{,i} + g_{ui}P^u_{,j}$ can then be written:

$$g_{ij,u}e^{au}P_a + P_{j,i} + P_{i,j} - P_a e^{al}[g_{jl,i} + g_{il,j}]$$

or

$$P_{j,i} - (1/2)P_a e^{al}(g_{jl,i} + g_{il,j} - g_{ij,l}) + P_{j,i} - (1/2)P_a e^{al}(g_{jl,i} + g_{il,j} - g_{ij,l}) = P_{i;j} + P_{j;i}$$

and therefore

$$\bar{g}_{ij,o} = g_{ij,o}Q + P_{i;j} + P_{j;i} + g_{oj}Q_{,i} + g_{oj}Q_{,j} \qquad (27)$$

A similar treatment for g and g gives

$$\bar{g}_{oo} = g_{uv}P^uP^v + 2g_{ou}QP^u + g_{00}Q^2$$

$$\bar{g}_{io} = g_{iv}P^v + g_{oi}Q$$

With these results it is a trivial matter to check the correctness of (10) (with Q = 1 and $P^i = -e^{ui} g_{uo}$), of (12) to (14), and the statement related to (20) with

$$Q = (-g_{oo})^{-1/2} \qquad ; \qquad P^i = -g_{jo} (-g_{oo})^{-1/2} e^{ij}$$

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